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Problem of Measurement Within the Operator Formulation of Hybrid Systems

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The basic concepts of classical mechanics are given in operator form. Then a hybrid systems approach with the operator formulation of both quantum and classical sectors is applied to the case of an ideal nonselective measurement. It is found that the dynamical equation, consisting of the Schrödinger and Liouville dynamics, produces noncausal evolution when the initial state of the measured (quantum mechanical) system and measuring apparatus–classical mechanical system is chosen to be as demanded in discussions regarding the problem of measurement. Nonuniqueness of possible realizations of the transition from a pure noncorrelated to a mixed correlated state is analyzed in detail. It is concluded that the state of the quantum mechanical system instantaneously collapses because of the nonnegativity of probabilities, and a dynamical model of this reduction is proposed.

1. INTRODUCTION

A correct theory of combined quantum mechanical and classical mechanical systems has to differ from quantum mechanics (QM) and classical mechanics (CM) with respect to causality and related topics. This is because the dynamical equations of QM and CM, taken separately, cannot lead to such changes of states that can happen in a (quantum) measurement processes. Quantum and classical mechanics are causal theories in which pure states can evolve, according to the appropriate equations of motion, only into pure states, not into mixed ones. For a process of nonselective measurement on a QM system done by an apparatus which is a CM system, there is a possibility for transitions from a pure to a mixed state.

An interesting approach to hybrid systems (consisting of one quantum and one classical system) has been proposed (Aleksandrov, 1981; Boucher

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and Trashen, 1988) which uses for states and observables the direct product of QM and CM representatives. The dynamical equation there introduced is, say, a superposition of QM and CM dynamical equations. It has been objected that this equation of motion does not save the nonnegativity of states, which has to be unaltered if the theory is to be physically meaningful. Otherwise, there would be events whose occurrence would be characterized by negative probabilities. However, we shall show that a hybrid systems approach (HSA) gives an adequate theoretical framework for description of an ideal nonselective measurement.

The strategy will be the following. First, for a particular choice of initial state of a QM system and measuring apparatus which addresses the problem of measurement, it will be shown that a correlated state, in contrast to the initial state, cannot be pure. Second, it will be found that the (dramatic) change of purity can be formally realized in more than one way; only one of them will be unphysical for the involved negative probabilities. In order to find what should be taken as the state of this hybrid system after the beginning of measurement, an analysis will show that the change of purity is necessarily followed by the change of this or that property of the state. Of course, we shall try to keep the argumentation physically grounded. Precisely, the necessary requirements of respecting the physical meaning whenever possible and/or considering only physically meaningful mathematical entities when physical problems are discussed will be sufficient for finding a physically meaningful possibility for the mixed correlated state. It will become obvious that this state is in accordance with the expected collapse of the QM state.

Before discussing the problem of measurement, we shall propose an operator formulation of classical mechanics. We shall use it instead of the standard phase space formulation of CM within the HSA since it will allow us to proceed with the argumentation in more complete way. However, it can be used separately for other purposes (Prvanović and Marić, 1999). Our proposal is almost the same as the one used by Sudarshan and co-workers (Gautam *et al.*, 1979; Shery and Sudarshan, 1978, 1979), while there are some similarities between our proposal and those given in Cohn (1980) and Sala and Muga (1994).

2. THE OPERATOR DESCRIPTION OF CLASSICAL MECHANICS

The most important features of the phase space formulation of classical mechanics are (1) the algebra of observables is commutative, (2) the equation of motion is the Liouville equation and it incorporates the Poisson bracket, and (3) pure states are those with sharp values of position and momentum, the values of which are, in general, independent. All these will hold for the operator formulation of CM, which we are going to introduce heuristically.

Let the pure states for position, in the Dirac notation, be $|q\rangle$. Similarly, for the momentum, we have $|p\rangle$. In quantum mechanics independence of states is formalized by the use of the direct product. These prescriptions suggest that pure classical states should be related somehow with $|q\rangle \otimes |p\rangle$. Consequently, the operator formulation of classical mechanics should be looked for within the direct product of two rigged Hilbert spaces, say $\mathcal{H}^q \otimes \mathcal{H}^p$. In such a space, one can define an algebra of classical observables. It is the algebra of polynomials in $\hat{q}_{cm} = \hat{q} \otimes \hat{I}$ and $\hat{p}_{cm} = \hat{I} \otimes \hat{p}$ with real coefficients, etc. The elements of this algebra are Hermitian operators and they obviously commute since $[\hat{q}_{cm}, \hat{p}_{cm}] = 0$. Further, one can define states as in the standard formulation of CM as functions of position and momentum, which are now operators. Precisely, one can define the pure states as

$$\begin{aligned}
&\delta(\hat{q} - q(t)) \otimes \delta(\hat{p} - p(t)) \\
&\quad = \int\int \delta(q - q(t))\delta(p - p(t))|q\rangle\langle q| \otimes |p\rangle\langle p|\, dq\, dp \\
&\quad = |q(t)\rangle\langle q(t)| \otimes |p(t)\rangle\langle p(t)| \qquad (1)
\end{aligned}$$

The pure and (noncoherently) mixed states, commonly denoted by $\rho(\hat{q}_{cm}, \hat{p}_{cm}, t)$ in this formulation, are nonnegative and Hermitian operators, normalized to $\delta^2(0)$ if, for the same function of real numbers, i.e., for $\rho(q, p, t)$, it holds that $\rho(q, p, t) \in \mathbf{R}$, $\rho(q, p, t) \geq 0$, and $\int\int \rho(q, p, t)\, dq\, dp = 1$. If one calculates the mean values of observables, e.g., $f(\hat{q}_{cm}, \hat{p}_{cm})$, in the state $\rho(\hat{q}_{cm}, \hat{p}_{cm}, t))$ by the Ansatz

$$\frac{\mathrm{Tr}(f(\hat{q}_{cm}, \hat{p}_{cm})\rho(\hat{q}_{cm}, \hat{p}_{cm}, t))}{\mathrm{Tr}\rho(\hat{q}_{cm}, \hat{p}_{cm}, t)} \qquad (2)$$

then it will be equal to the usually calculated $\int\int f(q, p)\rho(q, p, t)\, dq\, dp$, where $f(q, p)$ and $\rho(q, p, t)$ are the phase space representatives of corresponding observable and state, respectively. It is easy to see that, due to (1) and (2), the third characteristic of the phase space formulation holds for the new one as well.

For the criterion of purity we propose the idempotency of state, up to its norm. This criterion is obviously satisfied for (1) and it is adequate for the standard formulation of QM. Therefore, we shall use it for the operator formulation of hybrid systems, too.

The dynamical equation in the new formulation can be defined in accordance with feature 2 as

$$\frac{\partial\rho(\hat{q}_{cm}, \hat{p}_{cm}, t)}{\partial t} = \frac{\partial H(\hat{q}_{cm}, \hat{p}_{cm})}{\partial \hat{q}_{cm}} \frac{\partial\rho(\hat{q}_{cm}, \hat{p}_{cm}, t)}{\partial \hat{p}_{cm}} - \frac{\partial H(\hat{q}_{cm}, \hat{p}_{cm})}{\partial \hat{p}_{cm}} \frac{\partial\rho(\hat{q}_{cm}, \hat{p}_{cm}, t)}{\partial \hat{q}_{cm}} \tag{3}$$

For the RHS of (3) we shall use the notation $\{H(\hat{q}_{cm}, \hat{p}_{cm}), \rho(\hat{q}_{cm}, \hat{p}_{cm}, t)\}$.

The standard formulation of classical mechanics appears through the kernels of the operator formulation in the $|q\rangle \otimes |p\rangle$ representation. This, together with (2), can be used as the proof of equivalence of the two formulations. The other important remark is that, after the $\hat{q}_{cm}$ and $\hat{p}_{cm}$ have been defined, every other observable and every state can and have to be expressed as some function of just these two.

3. AN OUTLINE OF THE HYBRID SYSTEMS APPROACH

A physical system is called a hybrid system if it consists of one QM and one CM system. Such systems were discussed in Aleksandrov (1981), Anderson (1995), Boucher and Trashen (1988), Caro and Salcedo (1999), Diósi (1999), Peres (1993), Prezhdo and Kisil (1997), and Salcedo (1996). Instead of reviewing these articles with the purpose of introducing the formalism for hybrid systems, we shall start with the standard treatment of two QM systems and then, by substituting one quantum with one classical system, find directly the appropriate theoretical framework.

The standard formulation of two quantum systems needs the direct product of two (rigged) Hilbert spaces, say $\mathcal{H}_{qm1} \otimes \mathcal{H}_{qm2}$. The states of these systems evolve according to the Schrödinger equation with the Hamiltonian $\sum_\alpha \hat{H}^\alpha_{qm1} \otimes \hat{H}^\alpha_{qm2}$, for which

$$\begin{aligned}
\frac{\partial(\sum_{ij}\hat{\rho}^{ij}_{qm1}(t) \otimes \hat{\rho}^{ij}_{qm2}(t))}{\partial t}
&= \frac{1}{i\hbar}\left[\sum_\alpha \hat{H}^\alpha_{qm1} \otimes \hat{H}^\alpha_{qm2}, \sum_{ij} \hat{\rho}^{ij}_{qm1}(t) \otimes \hat{\rho}^{ij}_{qm2}(t)\right] \\
&= \sum_{\alpha ij} \frac{1}{i\hbar}[\hat{H}^\alpha_{qm1}, \hat{\rho}^{ij}_{qm1}(t)] \otimes \frac{\hat{H}^\alpha_{qm2}\hat{\rho}^{ij}_{qm2}(t) + \hat{\rho}^{ij}_{qm2}(t)\hat{H}^\alpha_{qm2}}{2} \\
&\quad + \sum_{\alpha ij} \frac{\hat{H}^\alpha_{qm1}\hat{\rho}^{ij}_{qm1}(t) + \hat{\rho}^{ij}_{qm1}(t)\hat{H}^\alpha_{qm1}}{2} \otimes \frac{1}{i\hbar}[\hat{H}^\alpha_{qm2}, \hat{\rho}^{ij}_{qm2}(t)]
\end{aligned} \tag{4}$$

With $\sum_{ij} \hat{\rho}^{ij}_{qm1}(t) \otimes \hat{\rho}^{ij}_{qm2}(t)$ (and more for the one in the next expression) we want to accommodate the notation for states to that type of correlation which will be discussed below.

Suppose now that the second system is classical. This means that everything related to this system in (4) has to be translated into the classical counterparts. Having in mind the above formulation of CM, we propose

$$\frac{\partial(\Sigma_{ij}\ \hat{\rho}^{ij}_{qm}(t) \otimes \hat{\rho}^{ij}_{cm}(t))}{\partial t}$$
$$= \sum_{\alpha ij} \frac{1}{i\hbar} [\hat{H}^{\alpha}_{qm}, \hat{\rho}^{ij}_{qm}(t)] \otimes \frac{\hat{H}^{\alpha}_{cm}\hat{\rho}^{ij}_{cm}(t) + \hat{\rho}^{ij}_{cm}(t)\ \hat{H}^{\alpha}_{cm}}{2}$$
$$+ \sum_{\alpha ij} \frac{\hat{H}^{\alpha}_{qm}\hat{\rho}^{ij}_{qm}(t) + \hat{\rho}^{ij}_{qm}(t)\hat{H}^{\alpha}_{qm}}{2} \otimes \{\hat{H}^{\alpha}_{cm}, \hat{\rho}^{ij}_{cm}(t)\} \tag{5}$$

as the dynamical equation. The explanation follows. The first system remained quantum mechanical, so its type of evolution is left unaltered. The Poisson bracket is there instead of the second commutator because classical systems evolve according to the Liouville equation. It is defined as in (3); the partial derivatives are with respect to the classical coordinate and momentum: $\hat{q} \otimes \hat{I}$ and $\hat{I} \otimes \hat{p}$. All states and observables, both QM and CM, appear in the operator form, i.e., the hybrid system is defined in $\mathcal{H}_{qm} \otimes \mathcal{H}^{g}_{cm} \otimes \mathcal{H}^{p}_{cm}$. (Note that the coordinate and momentum of the quantum and classical systems are operators acting in $\mathcal{H}_{qm}$ and $\mathcal{H}^{q}_{cm} \otimes \mathcal{H}^{p}_{cm}$, respectively.) Some justifications of (5) are given in due course.

Similar equations, in the c-number formulation of CM, were proposed in Aleksandrov (1981), Anderson (1995), Boucher and Trashen (1988), and Prezhdo and Kisil (1997). There one can find a variety of requirements that have to be imposed on the equation of motion for hybrid systems which will not be reviewed here. We just mention that the equation proposed in the first three of these references is antisymmetric, while the one in the last is not.

More discussions of this subject can be found in Caro and Salcedo (1999) and Salcedo (1996). The starting point there is that the formalism of hybrid systems should have all the mathematical properties of QM and CM (see Caro and Salcedo, 1999, for details) and it was concluded that such a formalism cannot exist. Rather than as a critique, we understand this result as an indication that the HSA is on the right track. Namely, we do not expect the appropriate formalism to possess all mathematical properties the same as in quantum and classical mechanics. On the contrary, we expect that the correct theory of hybrid systems will differ from these two mechanics with respect to the causality of evolution and, consequently, all other related topics. More precisely, in some cases the hybrid system equation of motion should lead to noncausal evolution. The example we have in mind, as we have mentioned, is the process of (quantum) measurement.

It has been objected that the HSA dynamical equation does not save the nonnegativity of states (Boucher and Trashen, 1988; Peres 1993; Prezhdo and Kisil, 1997). Our intention is to show, by a subtle analysis of the process of measurement, that this need not be so, i.e., the nonnegativity of states can be saved. After finding a dynamical equation as the source of noncausal evolution, which will be the case for (5), one needs to approach its solution through arguments that are of the same kind as those which qualify nonnegative states as meaningless, and to find acceptable states. This will become clear later. Here, we just mention that the noncausal evolution of the CM system alone occurs in the treatment of classical mechanics by the inverse Weyl transform of the Wigner function (see Muga and Snider, 1992, for details).

4. THE PROCESS OF MEASUREMENT

Usually, it is said that the measuring apparatus is a classical system. The formalism of hybrid systems becomes then the natural choice for the representation of the process of (quantum) measurement. We shall consider the nonselective measurement within the operator formulation of HSA by taking that the states of the measured QM system and the measuring apparatus evolve under the action of

$$H_{qm}(\hat{q}\otimes\hat{I}\otimes\hat{I},\hat{p}\otimes\hat{I}\otimes\hat{I}) + H_{cm}(\hat{I}\otimes\hat{q}\otimes\hat{I},\hat{I}\otimes\hat{I}\otimes\hat{p})$$
$$+V_{qm}(\hat{q}\otimes\hat{I}\otimes\hat{I},\hat{p}\otimes\hat{I}\otimes\hat{I})\cdot V_{cm}(\hat{I}\otimes\hat{q}\otimes\hat{I},\hat{I}\otimes\hat{I}\otimes\hat{p})$$

To simplify the expressions, we shall use $\hat{H}_{qm}\otimes\hat{I}_{cm}+\hat{I}_{qm}\otimes\hat{H}_{cm}+\hat{V}_{qm}\otimes\hat{V}_{cm}$ as the notation for this Hamiltonian. The measured observable is $\hat{V}_{qm}=\Sigma_i\, v_i|\psi_i\rangle\langle\psi_i|\otimes\hat{I}\otimes\hat{I}$. It is necessary that $[\hat{H}_{qm},\hat{V}_{qm}]=0$ because, if the quantum system before the measurement was in one of the eigenstates of the measured observable, say $|\psi_i\rangle$, then this system would not change its state during the measurement. Then, $\hat{H}_{qm}$ can be diagonalized in the same basis: $\hat{H}_{qm}=\Sigma_i\, h_i|\psi_i\rangle\langle\psi_i|\otimes\hat{I}\otimes\hat{I}$. For the CM parts of the Hamiltonian it is reasonable to assume that they do not cause periodic motion of the pointer. We shall not specify the Hamiltonian in more detail because we are interested only in discussions related to the form of the state after the beginning of measurement.

For the initial state of the quantum system we shall take the pure state $|\Psi(t_o)\rangle$ and for the pointer of the apparatus we shall take that initially it is in a state with sharp values of position and momentum, say q_o and p_o, so the state of the hybrid system at the moment when the measurement starts is $\hat{\rho}_{qm}(t_o)\otimes\hat{\rho}_{cm}(t_o)=|\Psi(t_o)\rangle\langle\Psi(t_o)|\otimes|q_o\rangle\langle q_o|\otimes|p_o\rangle\langle p_o|$. Of course, the problem of measurement demands that $|\Psi(t_o)\rangle$ be a superposition $\Sigma_i\, c_i(t_o)|\psi_i\rangle$.

Due to the interaction term in the Hamiltonian, the state of the composite system will become correlated—the CM parts of the state will depend somehow on the eigenvalues of $\hat{V}_{qm}$. Let us use the notation $\Sigma_{ij}\ \hat{\rho}^{ij}_{qm}(t) \otimes \hat{\rho}^{ij}_{cm}(t)$ in order to allow the analysis of the *a priori* possible situation in which the CM parts of the state can depend on two different eigenvalues of $\hat{V}_{qm}$. With this notation, and the above for the Hamiltonian, the dynamics of the measurement becomes can be represented by

$$\frac{\partial(\Sigma_{ij}\ \hat{\rho}^{ij}_{qm}(t) \otimes \hat{\rho}^{ij}_{cm}(t))}{\partial t}$$

$$= \sum_{ij} \frac{1}{i\hbar} [\hat{H}_{qm}, \hat{\rho}^{ij}_{qm}(t)] \otimes \hat{\rho}^{ij}_{cm}(t) + \sum_{ij} \frac{1}{i\hbar} [\hat{V}_{qm}, \hat{\rho}^{ij}_{qm}(t)] \otimes \hat{V}_{cm}\hat{\rho}^{ij}_{cm}(t)$$
$$+ \sum_{ij} \hat{\rho}^{ij}_{qm}(t) \otimes \{\hat{H}_{cm}, \hat{\rho}^{ij}_{cm}(t)\}$$
$$+ \sum_{ij} \frac{1}{2} (\hat{V}_{qm}\hat{\rho}^{ij}_{qm}(t) + \hat{\rho}^{ij}_{qm}(t)\hat{V}_{qm}) \otimes \{\hat{V}_{cm}, \hat{\rho}^{ij}_{cm}(t)\} \tag{6}$$

where $\hat{H}_{cm}$, $\hat{V}_{cm}$, and $\rho^{ij}_{cm}(t)$ are derived in the Poisson bracket with respect to $\hat{q} \otimes \hat{I}$ and $\hat{I} \otimes \hat{p}$ that act in $\mathcal{H}^q_{cm} \otimes \mathcal{H}^p_{cm}$.

The solution of this dynamical equation will represent the state of the hybrid system at $t > t_o$ and the search for it can start by noticing that the CM terms $\hat{\rho}^{ii}_{cm}(t)$ attached to the quantum mechanical terms with equal indices $\hat{\rho}^{ii}_{cm}(t)$ (which we shall call diagonal terms) are $\hat{\rho}^{ii}_{qm}(t) = |q_i(t)\rangle\langle q_i(t)| \otimes |p_i(t)\rangle\langle p_i(t)|$, where the indices in $|q_i(t)\rangle$ and $|p_i(t)\rangle$ underline the dependence on one eigenvalue of $\hat{V}_{qm}$. Guided by this dependence of each CM bra and ket of $\hat{\rho}^{ii}_{cm}(t)$ on one eigenvalue of $\hat{V}_{qm}$, as the candidate for the correlated state we shall consider the coherent mixture

$$\sum_{ij} c_{ij}(t)|\psi_i\rangle\langle\psi_j| \otimes |q_i(t)\rangle\langle q_j(t)| \otimes |p_i(t)\rangle\langle p_j(t)| \tag{7}$$

There are two other candidates for correlated state. The first is

$$\sum_{ij} c_{ij}(t)|\psi_i\rangle\langle\psi_j| \otimes |q_{ij}(t)\rangle\langle q_{ij}(t)| \otimes |p_{ij}(t)\rangle\langle p_{ij}(t)| \tag{8}$$

where the indices in $|q_{ij}(t)\rangle$ and $|p_{ij}(t)\rangle$ to represent the dependence on two eigenvalues of $\hat{V}_{qm}$ in the form $\frac{1}{2}(v_i + v_j)$. The same holds for $\langle q_{ij}(t)|$ and $\langle p_{ij}(t)|$. The motivation for this comes from the symmetrization of the QM sector in front of the Poisson bracket on the RHS of (5). The terms $\hat{\rho}^{ij}_{cm}(t)$ of (8) are diagonal with respect to the eigenbasis of $\hat{q}_{cm}$ and $\hat{p}_{cm}$ for each pair of indices, while these terms of (7) for $i \neq j$ are not. As the third candidate for correlated state we shall consider the noncoherent mixture

$$\sum_i |c_i(t_o)|^2 |\psi_i\rangle\langle\psi_i| \otimes |q_i(t)\rangle\langle q_i(t)| \otimes |p_i(t)\rangle\langle p_i(t)| \tag{9}$$

All three states have the same diagonal terms $\hat{\rho}_{qm}^{ii}(t) \otimes \hat{\rho}_{cm}^{ii}(t)$. The difference between these states is in the CM $i \neq j$ terms. Each ket and bra of $\hat{\rho}_{cm}^{ij}(t)$, $i \neq j$, in (7) depend on only one eigenvalue of $\hat{V}_{qm}$, in (8) they depend on two eigenvalues, and in expression (9) there are no such terms.

The state (7) is designed to represent as pure, nonnegative, and Hermitian correlated state as is the initial state and it has nondiagonal QM terms (with respect to the basis $|\psi_i\rangle$) as the state $|\Psi(t_o)\rangle\langle\Psi(t_o)|$. [The state is taken to be pure if it is idempotent up to the norm: $\hat{\rho}^2 = \delta^2(0) \cdot \hat{\rho}$.] The purity of (7) rests on the same type of time development (dependence on one v_i) as $|q_i(t)\rangle$ and $|p_i(t)\rangle$, regardless of whether they belong to $\hat{\rho}_{cm}^{ij}(t)$ with $i = j$ or with $i \neq j$. But the following holds. The initial state of the apparatus is diagonal with respect to the eigenbasis of $\hat{q}_{cm}$ and $\hat{p}_{cm}$. To "create" the nondiagonal terms from it in a form which ensures purity, one would need to introduce operators that do not commute with $\hat{q}_{cm}$ and $\hat{p}_{cm}$ to act on CM states. One would need to take some other dynamical equation instead of (5) as well. That dynamical equation should use the commutator for both subsystems, as is the case for (4). If one would do that, then, in a treatment of the apparatus, one would neglect the requirements 1 and 2 which are part of the definition of a classical system (see Section 2). This type of reasoning would be *à la* von Neumann's approach to the measurement process where the apparatus and measured system are both treated as quantum systems. Instead of going in that direction, we are considering here the apparatus as a classical system, defined in the above way. By this we avoid the well-known problems that arise with states such as (7). [According to (7), there could be a superposition of pointer states which is unobserved. Then, the problem of measurement, as we understand it, is to explain why and describe how a state similar to (7) collapses to a state similar to (9).]

A less descriptive and more formal way to look for a solution is to assume that the time dependence of the evolved state is as represented by (7). Then, by substituting (7) in (6) in order to verify this, we find a contradiction. Namely, the CM $i \neq j$ terms of (7) do not commute with $\hat{q}_{cm}$ and $\hat{p}_{cm}$ for $t \neq t_o$, so they are not functions of only these observables. The partial derivatives $\partial/\partial\hat{q}_{cm}$ and $\partial/\partial\hat{p}_{cm}$ from the Poisson bracket "annihilate" the CM nondiagonal elements of (7) for $t > t_o$ when they act on them. For instance,

$$\frac{\partial}{\partial\hat{q}} |q_i(t)\rangle\langle q_j(t)| = \frac{\partial}{\partial\hat{q}} \delta(\hat{q} - q_i(t)) \cdot \delta_{i,j} \tag{10}$$

$(t > t_o)$ and similarly for $|p_i(t)\rangle\langle p_j(t)|$ under the action of $\partial/\partial\hat{p}_{cm}$. Thus, for the CM $i \neq j$ terms of (7) the RHS of (6) vanishes for $t > t_o$, while the LHS is not equal to zero, by assumption.

Let us stop for a moment and make some remarks. An immediate consequence of the fact that (7) does not satisfy (6) is that the initial purity of the state is lost due to the established correlation. This is confirmed by consideration of (8) and (9). These two states do satisfy (6), but they are both mixed—they are not idempotent up to the norm: $\hat{\rho}^2 \neq \delta^2(0) \cdot \hat{\rho}$. This property is plausible for (9). For (8) it is enough to notice that in $\hat{\rho}^2$ there is, for example, a term $|\psi_i\rangle\langle\psi_i| \otimes |q_{ij}(t)\rangle\langle q_{ij}(t)| \otimes |p_{ij}(t)\rangle\langle p_{ij}(t)|$ which is not present in $\hat{\rho}$. Therefore, the hybrid system dynamical equation produces in this particular case a noncausal evolution: a pure noncorrelated state transforms into a mixed correlated state (which is to be found). This is the crucial difference between (5) and the Schrödinger and Liouville dynamics that appear within it.

One can convince oneself, by looking at (8) and (9), that purity is not the only property of the initial state that changes instantaneously at the moment when interaction begins. Obviously, there are no $i \neq j$ terms in (9), the meaning of which is that the QM part of (9), in contrast to the initial one, is diagonal with respect to the basis $|\psi_i\rangle$. On the other hand, the state (8) is not a nonnegative operator for all $t > t_o$, while the initial state is. For all states that are not nonnegative operators one can construct properties—events—that would be "found" with negative probabilities if they were to be measured. In order to construct such a property for (8), it is helpful to notice that the CM parts of the $i \neq j$ terms of (8) are regular states of CM systems; they are different from those with $i = j$ and are accompanied by the QM "states" with vanishing trace. (By regular we mean per se realizable since they are diagonal, and "states" stands here, and it would be better to stay in all similar cases, because they can only be interpreted as impossible.)

For the related negative probabilities, states which are not nonnegative operators should be qualified as meaningless and, since they appear in the HSA, there have been objections to its relevance for physics. In what follows, we show that these probabilities are not unavoidable here. In other words, our intention is to rehabilitate the HSA and this will manifest itself in finding formal support for a physically meaningful state (9) that should be taken as the solution, not the unphysical state (8). The arguments have to be in accordance with physics since experience leads one to be unsatisfied with (8) and, of course, the HSA is aimed at formalizing the behavior of physical systems. The first argument, based on the validity of (10), will continue the analysis of (7). The second discussion, concentrated on (8) and unrelated to (10), will again designate that (9) is the proper solution, but, in contrast to the first one, it will proceed in a more interpretational than formal manner.

Our insistence on (7) rests on the fact that one can look on it as a trial state. It is the perfect choice for a trial state because it has the same physically relevant characteristics as the initial state and it is equal to the initial state for $t = t_o$, i.e., for $t \to t_o$, (7) approaches the initial state without any change when these characteristics are considered. Moreover, the need for a trial state comes from the absence (to our knowledge) of some rule that would prescribe how to manage the change of idempotency. After being substituted on the RHS of the dynamical equation, the trial state will indicate the appropriate type of time transformation. Then, by minimal modifications of this state, intended to adapt it to that type, the desired correlated state will be found.

The RHS of (6) for the CM $i \neq j$ terms of (7) vanishes for all $t > t_o$ according to (10). Exclusively for $t = t_o$ the CM $i \neq j$ terms of (7) can be expressed as functions of only $\hat{q}_{cm}$ and $\hat{p}_{cm}$ since $q_i(t_o) = q_o$ and $p_i(t_o) = p_o$ for all i. Only for this moment does the RHS of (6) for the CM $i \neq j$ terms of (7) not vanish. Therefore, one concludes that the CM parts of $i \neq j$ terms have to be constant after the instantaneous change at t_o, i.e., instead of those of (7), the QM nondiagonal terms have to be coupled with the time-independent CM terms for all $t > t_o$. This is how the dynamical equation designates that (8) should not be taken as the solution. What one has to do if one wants to accommodate (7) to the deduced time independence of the CM $i \neq j$ terms is to take for these terms (for $t > t_o$) some operators that do not involve time. Then, in order to satisfy (6), the operators should not be expressible as some functions of (only available) $\hat{q}_{cm}$ and $\hat{p}_{cm}$. On the other hand, with these operators one should not change either the Hermitian character or the nonnegativity of the hybrid system state since nothing asks for that. The resulting state, of course, has to be impure because any change of the CM $i \neq j$ terms of (7) affects its idempotency. In this way, (9) will be obtained as the appropriate solution.

Having in mind the functions of $\hat{q}_{cm}$, $\hat{p}_{cm}$ and operators that do not commute with these two, one may not want to accept (10). For the sake of mathematical rigor, let us clarify this. The CM nondiagonal terms of (7) cannot be expressed as functions depending only on $\hat{q}_{cm}$ and $\hat{p}_{cm}$, but they can be expressed as functions of these two if, first, the number of operators available is increased and, second, there is noncommutativity among them. How these functions would look like depends on these new operators. Since there are neither motivations nor instructions for their introduction coming from physics, they can be introduced freely. More precisely, these operators do not represent anything meaningful and they need not enclose any known mathematical structure. For instance, $|q_i(t)\rangle\langle q_j(t)|$ can be expressed as $\exp[(1/a)(q_i(t) - q_j(t))\hat{\pi}]\delta(\hat{q} - q_j(t))$, where $\hat{\pi}$ is not to be confused with the CM momentum since it acts in $\mathcal{H}^q$, not in $\mathcal{H}^p$, and $\langle q|\hat{\pi}|q'\rangle = a\,\partial\delta(q - q')/\partial q$. Here, a can be anything, it need not to be equal to $-i\hbar$ as in quantum mechanics. Another (even more pathological) example is the following. Since the CM nondiagonal dyads do not commute with $\hat{q}_{cm}$ and $\hat{p}_{cm}$, they can be used as the new operators, e.g., $|q_i(t)\rangle\langle q_j(t)| = F(q_i(t))^{-1}F(\hat{q})|q_i(t)\rangle\langle q_j(t)|$.

This shows that these nondiagonal dyads can be expressed as functions depending on $\hat{q}_{cm}$, $\hat{p}_{cm}$, and uncountably many other arguments—all nondiagonal dyads, where F can be any function. With these two examples we wanted to justify the need to bound considerations of CM in operator form to functions of only $\hat{q}_{cm}$ and $\hat{p}_{cm}$ [see Shery and Sudarshan (1978), where this problem is approached in a different manner]. On the other hand, the need to discuss the purity of the state of the hybrid system gives rise to the need to consider nondiagonality (with respect to the basis $|q\rangle \otimes |p\rangle$) of the CM state. When these two meet in the dynamical equation with expressions like (10), nothing unusual is done: the derivation of an entity which is not a function of that with respect to which it is derived has zero as a result. If one says that the LHS of (10) is just defined by the RHS of (10), then it should be noticed that (10) does not contradict any of the calculational rules of CM and QM because in the standard formulation of classical mechanics there is no possibility for realization of nondiagonality, while in the standard formulation of quantum mechanics there is no necessity for restriction to commutativity. Anyhow, let us proceed by supposing that one is not willing to accept (10) and/or that one finds the given support for (9) as not sufficiently convincing.

Even without (10), one is not free of contradiction if (7) is taken to be the solution. Due to the symmetrization of the QM sector, on the RHS of (6), in front of the second Poisson bracket there are two eigenvalues of $\hat{V}_{qm}$ coming from $\hat{\rho}^{ij}_{qm}(t)$ $(i \neq j)$ of (7). Because of this, the assumption that each ket and bra of $\hat{\rho}^{ij}_{cm}(t)$ $(i \neq j)$ of (7) depends on only one eigenvalue of $\hat{V}_{qm}$ is contradicted. Thus, it seems that to introduce noncommuting operators in $\mathcal{H}^{q}_{cm} \otimes \mathcal{H}^{p}_{cm}$ and/or to slightly modify (6) would not be enough to avoid some contradiction connected to (7) when it is seen as the result of evolution. However, it is not our intention to go into these areas, which are beyond the purpose of this article.

After discarding (7), one concludes that each ket and bra of $\hat{\rho}^{ij}_{cm}(t)$ $(i \neq j)$ would depend on two eigenvalues of $\hat{V}_{qm}$ coming from $\hat{\rho}^{ij}_{qm}(t)$ $(i \neq j)$ for $t > t_o$ if there would be $\hat{\rho}^{ij}_{qm}(t)$ $(i \neq j)$ for all such times. Therefore, the most important step in solving the dynamical equation for the above Hamiltonian is to find what happens with the initial QM state at the moment when interaction begins. Then it will be almost a trivial problem to find the state of the hybrid system at latter times. Or, more precisely, in the presence of $|\psi_i\rangle\langle\psi_j|$ $(i \neq j)$ for $t > t_o$ one has a dilemma: (8) or (9), the meaning of which is that by the assumed linearity of evolution, in a case when it is noncausal, one excludes the physical meaning of the evolved state, and vice versa.

From this point, our strategy for defending the HSA from objections that it might be unphysical to show that one finds it unphysical only after one has previously decided to prefer formal, rather than physical arguments and, moreover, only after one has neglected statements (being, by the way, of the same sort as those used for disqualification) that lead to physically meaningful states. Let us be more concrete. To find (8) it was necessary to start with the more formal assumption that the nondiagonality of the QM part of the state, with respect to the eigenbasis of $\hat{H}_{qm}$ and $\hat{V}_{qm}$, has not changed at the moment when the purity of the state has changed. Opposite to this is to assume that the diagonality of the QM part of the state with respect to the basis which is privileged at that time has not changed. Before the moment t_o, the QM part of the state is diagonal with respect to the eigenbasis of that observable for which $|\Psi(t_o)\rangle$ is the eigenstate. Only this basis can be characterized as privileged for that time because the corresponding observable has been used for preparation. For physics, each other basis, including the eigenbasis of $\hat{H}_{qm}$ and $\hat{V}_{qm}$, is less important, i.e., their significance comes from mathematics, not from physics—they can be used just to express the same state in different manners. After the moment t_o the privileged basis is the eigenbasis of $\hat{V}_{qm}$ (and $\hat{H}_{qm}$) because this observable is measured. So, instead of claiming that the nondiagonality with respect to the basis which is going to become privileged should not change, one can claim that the diagonality with respect to the actually privileged basis should not change. These statements express two different types of reasoning: the first one concentrates on the formal aspect of the operators representing states [leading to (8)], while the other one is concerned about the meaning [leading to (9)].

If the mentioned nondiagonality of the QM part of the initial state has survived t_o, then, according to (6), there would be CM systems in (realizable) states $\hat{\rho}^{ij}_{cm}(t)$ coupled to the QM nondiagonal terms, as is given by (8). But, the probability of event $\hat{I} \otimes |q_{ij}(t)\rangle\langle q_{ij}(t)| \otimes |p_{ij}(t)\rangle\langle p_{ij}(t)|$ for the state (8) is equal to zero for all $t > t_o$, where $i \neq j$. Neither apparatus would be in any of the states $\hat{\rho}^{ij}_{cm}(t)$ with $i \neq j$ after the beginning of measurement. (This is not the case for $i = j$.) So, if the statements about probability are of any importance, before proclaiming (6) as inadequate because it does not save the nonnegativity of the initial state, one should accept that in the states $\hat{\rho}^{ij}_{cm}(t)$ $(i \neq j)$ neither apparatus would be. The consequence of this is that the assumption of surviving QM nondiagonal terms is not correct. In physics, where the probability is a significant concept, what has just been found is enough to conclude that (9) should be taken as the solution. Simultaneously, by finding that (8) is unphysical, one finds why it is so: it is unphysical because some states of CM systems that are not exhibited by any apparatus are kept in the representation of the state of the hybrid system. By taking this into account, i.e., by reexpressing (8) with this in mind, one will find (9) as the proper state of the hybrid system.

Finally, the validity of the hybrid system dynamical equation can be verified for situations for which it is easy to say what behavior is desired.

For example, the hybrid system dynamical equation gives the standard one-to-one evolutions of QM and CM subsystems when the interaction term in the Hamiltonian is absent. In this case the evolved states are of the same purity and nonnegativity as the initial states. Moreover, for the above Hamiltonian and the initial state of the hybrid system $\Sigma_i |c_i(t_o)|^2 |\psi_i\rangle\langle\psi_i| \otimes |q_o\rangle\langle q_o| \otimes |p_o\rangle\langle p_o|$, the evolved state is not unphysical, it is (9). These examples justify the hybrid system dynamical equation as the proper one. So, it is likely that this holds for the case addressing the problem of measurement.

5. CONCLUDING REMARKS

Without an operator formulation of classical mechanics, the analysis of the problem of measurement in the hybrid systems approach would not be complete. First, this formulation enabled us to consider the pure correlated state and then, after finding that such a state cannot satisfy the dynamical equation, to conclude that this dynamical equation produces noncausal evolution: when the pure initial state of the quantum system is not an eigenstate of the measured observable, the initial state of the hybrid system, which is also pure, necessarily and instantaneously transforms into a mixed correlated state. Second, when it was not so obvious how the dynamical equation should be solved, the operator formulation offered support for one particular way.

The choice of a state representing the hybrid system after the beginning of measurement is important since the appropriateness of the HSA for physics depends on it. Both states that satisfy the dynamical equation for the given Hamiltonian are same regarding the impurity and absence of CM $i \neq j$ terms, so the essential part of the physical meaning is one and the same. Only the way of expressing these differs from (8) to (9). For their properties, perhaps it would not be wrong to say that (9) is the physical result of hybrid systems dynamics and that (8) is a physically unacceptable mathematical solution.

The third way in which the operator formulation of classical mechanics is useful is that it allows one to design, say, a dynamical model of instantaneous decoherence. Namely, in the resulting proposal of HSA, the partial derivations in the Poisson bracket change the CM nondiagonal terms at t_o [if the initial state is seen as (7) with $t = t_o$] and then obstruct their further time development according to (10), i.e., these derivations annihilate the CM nondiagonal terms. So, in this proposal, the dynamics is the cause of collapse. The reduction of the quantum mechanical state is the consequence of the disappearance of classical mechanical $i \neq j$ terms. The part of the interpretation of (8) which is meaningful from the point of view of everyday experience has led to the same conclusion: terms $\hat{\rho}^{ij}_{qm}(t)$ vanish because to them related and per se realizable events $\hat{\rho}^{ij}_{cm}(t)$ cannot occur. In another words, the reason for the decoherence of the QM state in case of a measurement lies in the Liouville equation. It is linear only in probability densities within the framework of commutative operators that represent position and momentum of classical systems, in contrast to the Schrödinger equation, which is linear in both the probability densities and the probability amplitudes.

In almost the same manner as the action of projectors describes measurement in standard quantum mechanics, the action of partial derivatives does it here. If one compares the standard formulation of QM and the operator formulation of HSA, one finds them similar, for they treat decoherence-collapse as an instantaneous process. They differ since decoherence is dynamical here. The operator formulation of HSA in this way answers one question raised in quantum mechanics: How should the collapse be described? But there is another, more important question: Why does it happens? The hybrid system approach does not ask for some ad hoc concepts to explain the collapse of the state; the nonnegativity of probabilities is enough. Because of the nonnegativity of probabilities, the collapse of the state is the only possible way of evolution for physical systems in the considered case and it is as ordinary as the one-to-one evolutions are in other cases. If one wants to stay within the single Hilbert space formulation of QM, then the HSA puts the projection postulate on a more solid ground. It is not related to the consciousness of the observer, but to the nonnegativity of probabilities.

The nonnegativity of probabilities is, and should be, incorporated among the first principles of any physical theory. The hybrid system approach differs from classical and quantum mechanics only in that this principle is invoked not just at the beginning, when the initial state is represented, but for the moments at which states lose purity as well. This rule offers an alternative in the search for a solution and it is not in contradiction with these two mechanics. There are no such moments when only the Schrödinger or the Liouville equation is solved within the Hilbert space and phase space, respectively, so there is no rule which would be contradicted. If it is represented (as some kind of superselection rule) in $\mathcal{H}_{qm} \otimes \mathcal{H}_{cm}^{q} \otimes \mathcal{H}_{cm}^{p}$ as a restriction to consider only states that are nonnegative operators, then there would be only two possibilities for a correlated state in the analyzed case: the coherent mixture (7) and the noncoherent mixture (9). The state (9) would follow immediately after finding that (7) cannot satisfy the equation of motion. (There is strong similarity between this and the way of solving the Maxwell quations where only the physical meaningful solution is retained.)

Roughly speaking, the procedure of solving differential equations consists in two steps. The first one is to find all functions that satisfy it (if there is any) and the second is, if there are more than one function, to select one by imposing some condition. The most often used is the Cauchy condition. Adapted to the present framework, it reads: The state at later times is the one which for $t = t_o$ becomes equal to the initial state. With this condition one wants to express the assumed continuity of evolution. The state (8) obviously follows in this way and, since this state is unphysical, the HSA shows that the state of physical systems in the considered case has to evolve discontinuously. From our point of view, this strongly recommends the HSA for a theory of combined classical and quantum systems.

The objections addressing the relevance of the HSA for physics are closely related to the application of the Cauchy condition in, let us say, a careless manner. We believe that it is not correct to take it as the unique supplementary condition and that it is not appropriate to impose it without noticing that something dramatic happens with the initial state at the moment when the evolution begins. If one were to disregard the unavoidable change of purity of the initial state and treat it as unimportant, then one would lose the physics from the very beginning. Moreover, then one could not discuss the physical meaning of the solution at the end because it would make such a consideration inconsistent. Only after noticing that [according to the discussion based on (7) and (10)] the initial state has changed instantaneously and discontinuously and after finding in which state it has changed should one apply the Cauchy condition, for then it is adequate because the further evolution is causal and in all aspects continuous. If this, the rule to invoke the nonnegativity of probability for the moments at which states lose purity, and (10) are new at all, these rules are the slightest possible modifications of the previously used ones. Or perhaps they are just the accommodation of standard rules to new situations.

Needless to say, the state (9) is in agreement with what is usually expected to happen when the problem of measurement is considered in an abstract and ideal form. To each state of the measured quantum system, which are the eigenstates of the measured observable, there corresponds one pointer position and momentum. The *i*th eigenvalue of the measured observable occurs with probability $|c_i(t_o)|^2$, and, as was said, (9) takes place immediately after the apparatus in the state $|q_o\rangle \otimes |p_o\rangle$ has started to measure $\hat{V}_{qm}$ on the system in the pure state $|\Psi(t_o)\rangle$ which can be seen as $\Sigma_i\, c_i(t_o)|\psi_i\rangle$.

Once noticed, the departure from strict causality would also be noticed in (all) other aspects as some strange feature. For example Caro and Salcedo (1999) found that so-called universal privileged times in dynamics of hybrid systems appear. Here, t_o is such a moment. In contrast to their opinion, we believe that this is a rather nice property of the approach. Namely, for the described process, and all others that can be treated in the same way, pure states, can evolve into noncoherent mixtures, while noncoherent mixtures cannot evolve into coherent mixtures—pure states, i.e., when the nonnegativity of probability is respected, such processes are irreversible. This means that for them the entropy can only increase or stay constant. Then the distinguished moments of the increase of entropy can be used for defining an arrow of time.

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